

DEFINITIONS, METRICS, AND ALGORITHMS FOR DISPLACEMENT, JITTER, AND STABILITY

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Abstract

This paper introduces pointing error definitions and metrics that are mathematically well-founded and are meaningful to image quality and to other types of measurements. The definitions and metrics are accuracy, displacement, jitter, stability, and windowed stability. The metrics are time-domain and equivalent frequency-domain formulas. The frequency-domain formulas are particularly easy to apply. Useful metrics for displacement and jitter were introduced a decade ago but have not been utilized in requirements documents, and so they are reintroduced here. New definitions and metrics for stability and windowed stability are introduced in this paper.

INTRODUCTION

Specifications for pointing jitter and stability are among the most important for the design of a spacecraft attitude control system. But in most requirements documents they are also the most ambiguous and are often the subject of endless debate. This is because jitter and stability are usually ill defined, as clearly illustrated by examples in [1]. The purpose of this paper is to define clear, valid, and mathematically well founded pointing error definitions and metrics that will ensure acceptable performance of payload instruments that can be achieved by design and verified by analysis. Although attitude control systems engineers are generally not instrument experts, and vice versa, pointing error metrics must be defined in a way that is meaningful to both. This mutual understanding is fostered by the introduction of some simple concepts from optics that demonstrate the effect of line-of-sight motion on image quality. The pointing error metrics defined in this paper are related directly to image quality and are useful for other types of measurements. These metrics are accuracy, displacement, jitter, stability, and windowed stability. These are expressed as time-domain and equivalent frequency-domain formulas. The frequency-domain formulas are particularly easy to apply and are propitious in attitude control system design and analysis.

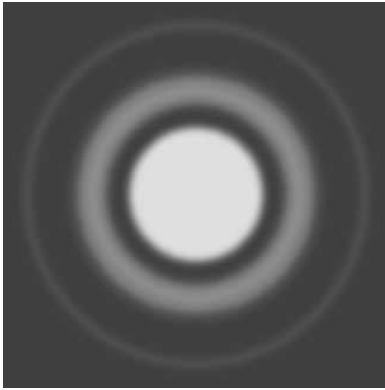
Some simple concepts from optics are introduced in the next section to demonstrate the effect of line-of-sight motion on imaging quality and to justify our definition of jitter. Definitions of pointing accuracy, displacement, jitter, and stability are then stated followed by the corresponding mathematical descriptions. Some aspects of pointing error specifications are presented and we discuss how to apply the mathematical descriptions and formulae.

The mathematical descriptions (metrics) for displacement and jitter were developed at the Jet Propulsion Laboratory circa 1990–1992 and reported in [1] and [2]. An example of their application in attitude control system design was reported in [3] and more recently in [4]. Related work is reported in [5, 6]. The displacement and jitter metrics have not become vogue, and so they are reintroduced in this paper with all credit given to the authors of [1] and [2]. Although [1]

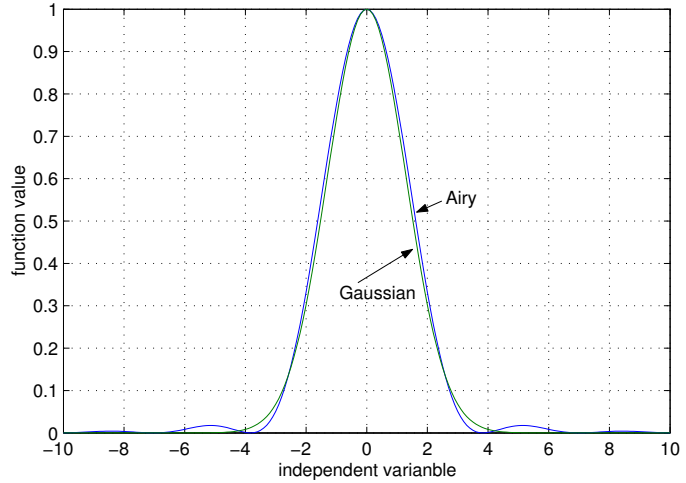
and [2] are a major contribution to the aerospace community, the authors unfortunately used the words “stability” and “jitter” interchangeably. These are distinguished in this paper with the introduction of new definitions and metrics for stability and windowed stability. The accuracy, jitter, and windowed stability metrics were recently approved for the STEREO mission pointing specification [7].

SOME CONCEPTS FROM OPTICS

The optical quality of an imaging system is limited partly by the Point Spread Function (PSF) of the optics. The function PSF $J(x)$, where x is spatial distance, describes how a column of light is spread out after passing through the optics. The PSF is no smaller than the Fraunhofer diffraction [8, 9], which is due to the aperture of the optics. The resolution of the charge-coupled device (CCD) detector array is the other major limiting component. The Airy Disc, which is illustrated in Figure 1a is the central bright region of the PSF. As illustrated in Figure 1b, the PSF can often be well approximated by a Gaussian function $G(x, \sigma_{\text{psf}}^2)$, which is normalized such that the diameter of the first dark ring around the Airy disc is unity. At the diffraction limit we have $\sigma_{\text{psf}} = 0.18$.



(a) Airy Disc



(b) Airy function and Gaussian approximation

Figure 1: Point Spread Function

The Full-Width Half-Maximum (FWHM) of the PSF is the diameter where $G(x, \sigma_{\text{psf}}^2) = 0.5$, so we have $\text{FWHM} = 2.35\sigma_{\text{psf}}$. The sensor’s response to alternating white and black lines has maximum and minimum intensities I_1 and I_2 . *Modulation* is given by $(I_1 - I_2)/(I_1 + I_2)$. *Resolution* is the smallest detail that can be distinguished in an image. It is determined by the number of line pairs of unit width such that a modulation of 0.5 is achieved. The width of a single line at the resolving limit equals the FWHM. Resolution is limited also by the spatial sampling of the CCD to approximately 2 pixels.

The effect of jitter motion on image quality can be determined by examining how it affects the point spread function.¹ In the presence of line-of-sight (LOS) motion $\Theta(t)$, the effective PSF is

$$J_e(x) = \mathcal{E}\{J(x)\} = \int_{-\infty}^{\infty} J(x - \Theta) \mathbf{p}(\Theta) d\Theta \quad (1)$$

¹The connection between pointing error and image quality, in particular how jitter widens the point spread function of an optical system, was introduced in [2].

where $\mathbf{p}(\Theta)$ is the probability density function of the LOS motion $\Theta(t)$. For Gaussian random motion, $\mathbf{p}(\Theta) = G(\Theta, \sigma_j^2)$, so we obtain²

$$J_e(x) = G(x, \sigma_{\text{psf}}^2 + \sigma_j^2) \quad (2)$$

The effective FWHM is then

$$\text{FWHMe} = 2.35(\sigma_{\text{psf}}^2 + \sigma_j^2)^{1/2} \quad (3)$$

The LOS motion over an exposure interval T is called *jitter* and the relevant metric is the *jitter variance* $\sigma_j^2(T)$. Thus we see from (3) how jitter reduces the resolution of an optical system: the variance of the jitter and the variance of the PSF add, thereby effectively widening the PSF.

POINTING DEFINITIONS

Figure 2 is a graph of attitude error versus time and shows three exposure intervals. A charge-coupled device (CCD) detector array integrates (gathers) photons during each exposure, and the exposures are separated by a readout interval, during which the image data is read from the CCD. The mean attitude over each interval is indicated by the horizontal line in each rectangle. The dot on this line is the mean attitude at the center of integration time. Blur of the image depends on the attitude motion during the exposure time. For image registration (*i.e.*, alignment of one image with another), we want some way to specify and measure the change in displacement of each image. The displacement of the image in the field of view is often not critical. However, in the SECCHI instrument on the STEREO spacecraft [7], it is critical to maintain an occulter on the solar disk to prevent excessive stray light from entering the optics so that the solar corona can be imaged, so displacement during an exposure must be kept small.

The concepts illustrated in the preceeding example are now made more concrete with the following verbal definitions:

Pointing Error is the angular rotation from the desired pointing direction to the actual pointing direction, or somewhat more loosely it is the difference between the actual pointing direction and the desired pointing direction.

Accuracy is the root-mean-square (RMS) pointing error of the line of sight (LOS) over any interval of time (window width $T_a \rightarrow \infty$). Rotation about the LOS may also be considered. The three body axes may be considered rather than the LOS. The RMS accuracy is designated $(\sigma_a^2 + \mu^2)^{1/2}$.

Displacement is the average pointing error of the LOS *within* an interval of T_d seconds. Generally T_d is the jitter window T_j defined below. Since displacement can change from one interval to another, the relevant measure is the RMS displacement $(\sigma_d^2 + \mu^2)^{1/2}$. Displacement is usually a small fraction (1% to 10%) of the field of view (FOV) of the instrument; the exact location of the target in the FOV is usually not important. In some applications, however, it is essential for other reasons to limit the displacement to a much smaller value.

Jitter is the RMS pointing error of the LOS *within* an interval of T_j seconds. This is usually specified to limit blur or streaking of an image during the exposure time, to limit signal variation in a sensor, or to limit some other mensuration (measurement) error. The RMS jitter is denoted σ_j .

²Non-Gaussian motion and motion dominated by sinusoidal disturbance was considered in [2].

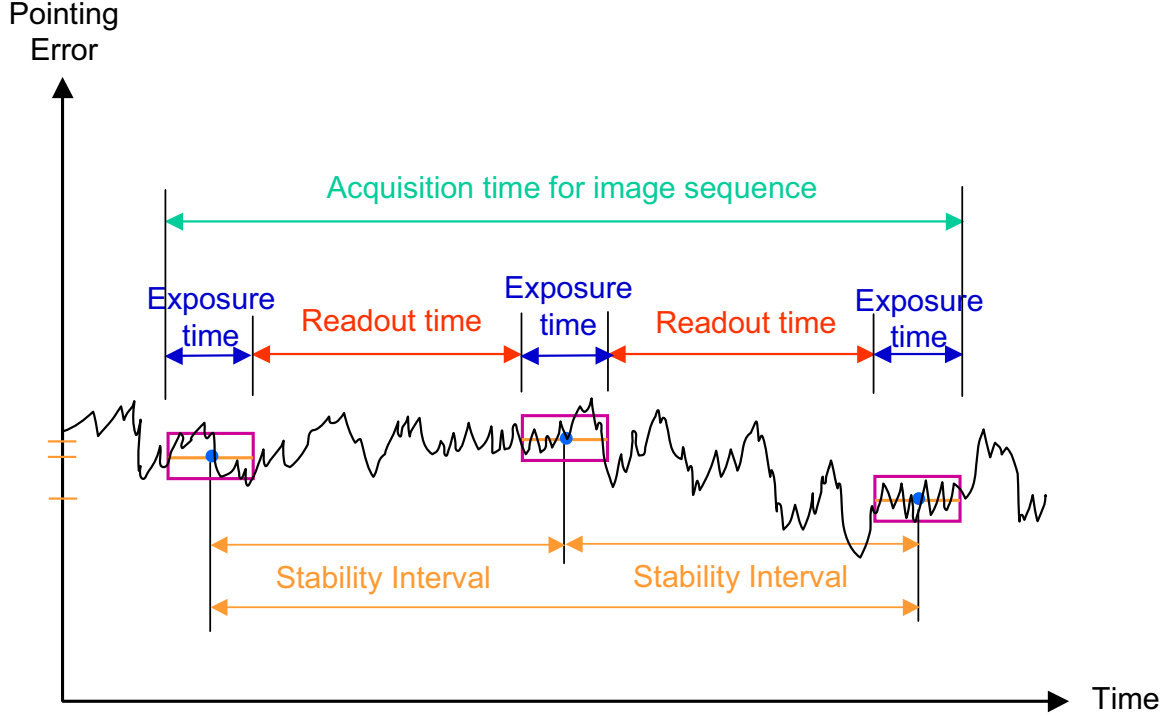


Figure 2: An illustration of pointing error showing three imaging windows.

Stability is the RMS *change* in the LOS from one end of an interval of T_s seconds to the other. This is usually specified for image registration or for data correlation from one time to another. The RMS stability is denoted σ_s .

Windowed Stability is the RMS change in displacement from the center-of-integration time of one measurement to the center-of-integration time of another measurement, where the displacement is the average attitude within a window of width T_d . The time between the center-of-integration times is designated T_s . The windowed stability σ_{sw} therefore depends on both T_d and T_s .

Multiple window times may be specified for various instruments. We may now proceed with mathematical statements of these definitions, which collectively are called pointing metrics. Because each metric is defined in terms of mean square error in the time domain, we have for each time domain expression an equivalent frequency domain expression via the Fourier transform.

POINTING METRICS

The pointing metrics presented below are mathematical expressions from which one can compute the mean square accuracy, displacement, jitter, stability, and windowed stability. These metrics are expressed in both the time domain and in the frequency domain. These metrics are summarized in Table 1 and their computational form is in Table 2.

Accuracy Metric

In the time domain, the accuracy metric is given by the expected value of the mean square attitude error $\Theta(t)$. The accuracy metric is thus given by³

$$\sigma_a^2 + \mu^2 = \mathcal{E}\{\Theta^2(t)\}. \quad (4)$$

where $\Theta(t)$ is the attitude (pointing) error. Note that the mean is not subtracted, so the total error metric is the variance σ_a^2 plus the mean square μ^2 . (The mean and variance are the standard definitions $\mu = \mathcal{E}\{\Theta(t)\}$ and $\sigma_a^2 = \mathcal{E}\{(\Theta(t) - \mu)^2\}$.) Assuming that $\Theta(t)$ is ergodic, the accuracy metric may be computed as a time average,⁴

$$\sigma_a^2 + \mu^2 = \langle \Theta^2(t) \rangle. \quad (5)$$

In the frequency domain, the accuracy metric is given by

$$\sigma_a^2 + \mu^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) d\omega, \quad (6)$$

where $S_{\Theta}(\omega)$ is the power spectral density of $\Theta(t)$. The time-domain and the frequency-domain expressions are mathematically equivalent.

Displacement Metric

In the time domain, the displacement (or shift) metric is the expected value of the squared average pointing error within a window of width T . The displacement metric is given by

$$\sigma_d^2(T) + \mu^2 = \mathcal{E}\{\langle \Theta(t) \rangle_T^2\}. \quad (7)$$

In the frequency domain, the displacement metric is

$$\sigma_d^2(T) + \mu^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_d(\omega T) d\omega, \quad (8)$$

where W_d is the **displacement weighting function** given by⁵

$$W_d(\omega T) = \frac{2(1 - \cos \omega T)}{(\omega T)^2}. \quad (9)$$

The displacement weighting function is plotted in the Figure 3. It approaches unity as $\omega \rightarrow 0$ and goes to zero as $\omega \rightarrow \infty$. Thus, low-frequency attitude motion contributes to displacement more than high-frequency motion.

³The notation $\mathcal{E}\{\cdot\}$ denotes the expected value of its argument and $\langle \cdot \rangle_T$ denotes the average of its argument over a time interval of length T .

⁴The time average over a window of width T seconds is given by $\langle x(t) \rangle_T = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\xi) d\xi$. When the T is omitted from $\langle \cdot \rangle_T$ we mean that the average is computed with $T \rightarrow \infty$. In the discrete-time domain where $x_i = x(t_i)$, the time average is simply $\langle x_i \rangle_N = \frac{1}{N} \sum_{i=1}^N x_i$ where N is the number of samples within the window of width T . Interpolation may be necessary if T is not an integral multiple of the sample rate.

⁵This equation could be written $W_d(\nu) = \left(\frac{\sin \nu}{\nu} \right)^2$, but it does not appear this way in the literature in this context, and we will not attempt to change the established convention.

Jitter Metric

In the time domain, the jitter metric is given by the expected value of the mean square motion within a window of width T , where the mean is a time average. The jitter metric is thus given by

$$\sigma_j^2(T) = \mathcal{E}\{\langle (\Theta(t) - \langle \Theta(t) \rangle_T)^2 \rangle_T\} \quad (10a)$$

$$= \mathcal{E}\{\langle \Theta^2(t) \rangle_T - \langle \Theta(t) \rangle_T^2\} \quad (10b)$$

$$= \sigma_a^2 - \sigma_d^2(T) . \quad (10c)$$

In the frequency domain, the jitter metric is

$$\sigma_j^2(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_j(\omega T) d\omega , \quad (11)$$

where W_j is the **jitter weighting function** given by

$$W_j(\omega T) = 1 - \frac{2(1 - \cos \omega T)}{(\omega T)^2} . \quad (12)$$

The jitter weighting function is plotted in the Figure 3. It reaches its first peak at $\omega = 2\pi/T$ rad/sec. Attitude error at frequencies greater than $1/T$ (Hz) contribute fully to jitter (their weighting is essentially 1) whereas attitude error at lower frequencies are weighted less than 1. Large amplitude attitude error at frequencies less than $1/T$ Hz can contribute significantly to jitter even though their weighting is less than 1.

A useful fact is that $\sigma_j^2(\tau) \leq \sigma_j^2(T)$ for all windows $\tau < T$ because the graph of $W_j(\omega\tau)$ moves to the right as τ decreases, thereby weighting the attitude error spectrum less at lower frequencies. Actually this is only essentially true; $\sigma_j^2(\tau)$ does not quite decrease monotonically as τ decreases (due to the wiggles in W_j), but any temporary increase is negligible for all practical purposes.

Note that the root-sum-square of the displacement and jitter requirements, each with the same window width T , has to be less than or equal to the accuracy requirement. This is because by definition we have $\sigma_a^2 + \mu^2 = \sigma_d^2(T) + \mu^2 + \sigma_j^2(T)$, and each term has to satisfy the accuracy, displacement, and jitter requirements.

Non-Windowed Stability Metric

The non-windowed stability metric measures the mean square change in attitude from one instant to another. The change in attitude over an interval of length T is given by

$$\Delta_T(t) = \Theta(t) - \Theta(t - T) . \quad (13)$$

The stability metric is the mean square change in attitude given by

$$\sigma_s^2(T) = \mathcal{E}\{(\Delta_T(t))^2\} \quad (14a)$$

$$\begin{aligned} &= \mathcal{E}\{(\Theta(t) - \Theta(t - T))^2\} \\ &= 2[R(0) - R(T)] , \end{aligned} \quad (14b)$$

where $R(T)$ is the autocorrelation function of $\Theta(t)$. These equations suggest two ways of computing $\sigma_s^2(T)$ in the time-domain, either by a time average or by way of autocorrelation. In the frequency domain, the stability metric is

$$\sigma_s^2(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_s(\omega T) d\omega , \quad (15)$$

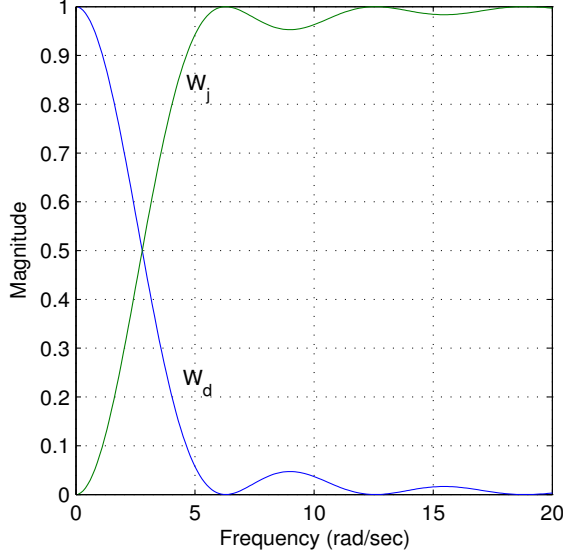


Figure 3: Displacement and Jitter Weighting Functions Compared

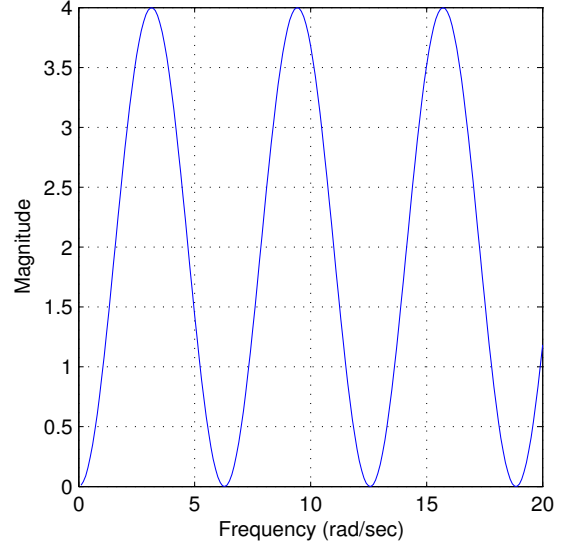


Figure 4: Stability Weighting Function

where W_s is the **stability weighting function** given by

$$W_s(\omega T) = 2(1 - \cos \omega T) . \quad (16)$$

A useful fact is that $W_s(\omega T) \leq 4$, so $\sigma_s^2(T) \leq 4(\sigma_a^2 + \eta^2)$. Therefore if $2(\sigma_a^2 + \eta^2)^{1/2}$ is less than the stability requirement, no further analysis of stability is needed. A less conservative test is $\sigma_s^2(T) \leq 4\sigma_a^2$.

Windowed Stability Metric

An obvious characteristic of the non-windowed stability weighting function $W_s(\omega T)$ is that it does not roll off at high frequency. This is because $\Theta(t)$ and $\Theta(t-T)$ are at instantaneous points in time. For measurements obtained by averaging or integrating over an interval τ (where, in general, $\tau = T_j$, the jitter window), we want to consider the displacements $\bar{\Theta}(t)$ and $\bar{\Theta}(t-T)$ at the centroids of the measurement intervals where

$$\bar{\Theta}(t) = \langle \Theta(t) \rangle_\tau = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} \Theta(\xi) d\xi . \quad (17)$$

The change in displacement over an interval of length T is given by

$$\bar{\Delta}_T(t) = \bar{\Theta}(t) - \bar{\Theta}(t-T) . \quad (18)$$

The Windowed Stability Metric (W-Stability) is the mean square change in displacement given by

$$\sigma_{sw}^2(T, \tau) = \mathcal{E}\{(\bar{\Delta}_T(t))^2\} \quad (19a)$$

$$\begin{aligned} &= \mathcal{E}\{(\bar{\Theta}(t) - \bar{\Theta}(t-T))^2\} \\ &= 2[R_\tau(0) - R_\tau(T)] , \end{aligned} \quad (19b)$$

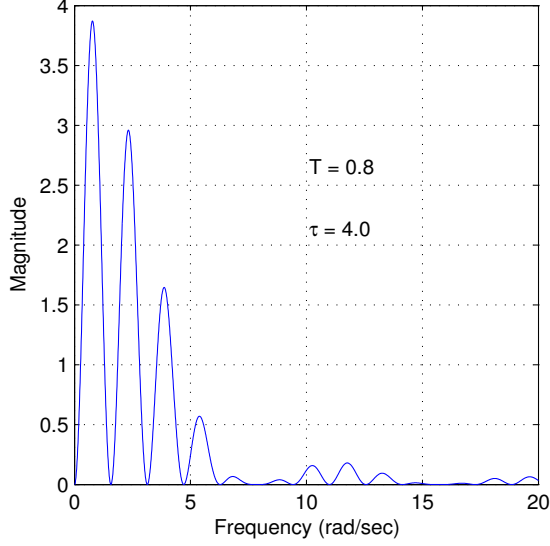


Figure 5: Windowed Stability Weighting Function

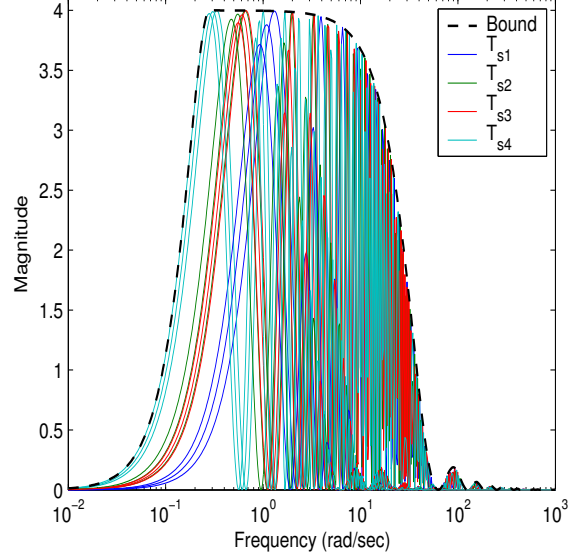


Figure 6: Over-bound on Four Variable Windowed Stability Weighting Functions

where $R_\tau(0) = \sigma_d^2(\tau) + \mu^2$ is the displacement metric given by equation (7) or (8) and $R_\tau(T) = \mathcal{E}\{\bar{\Theta}(t)\bar{\Theta}(t-T)\}$ is the autocorrelation function of $\bar{\Theta}(t)$. Both autocorrelation functions depend on τ , which is denoted by the subscript. The correlation $R_\tau(T)$ is expressed in the frequency domain by [10]

$$R_\tau(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\Theta(\omega) e^{j\omega T} W_d(\omega\tau) d\omega. \quad (20)$$

Substituting equations (8) and (20) into equation (19b) yields the Windowed Stability Metric

$$\begin{aligned} \sigma_{sw}^2(T, \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\Theta(\omega) 2(1 - e^{j\omega T}) W_d(\omega\tau) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\Theta(\omega) W_s(\omega T) W_d(\omega\tau) d\omega. \end{aligned} \quad (21)$$

We have made use of the fact that $S_\Theta(\omega)$ and $W_d(\omega\tau)$ are even function in order to replace the complex exponential with the real cosine. The **Windowed Stability Weighting Function** is thus given by

$$W_{sw}(\omega T, \omega\tau) = W_s(\omega T) W_d(\omega\tau). \quad (22)$$

The windowed stability weighting function is illustrated in Figure 5. The presence of $W_d(\omega\tau)$ in (21) results in the attitude error spectrum being weighted low at high frequency so that high-frequency components contribute little to the windowed stability variance.

In an imaging system, the exposure time (the jitter window T) may be variable. The time between images therefore varies, so the stability window time τ depends on T . It may not be possible or practical to evaluate the windowed stability variance over a continuum or grid of times. In this situation, the windowed stability weighting can be over-bounded by the simpler but more

conservative function

$$W_{\text{sw}}(\omega T, \omega \tau) \leq W'_{\text{sw}}(\omega T_{\text{max}}, \omega \tau_{\text{min}}) = \begin{cases} W_s(\omega T_{\text{max}}) & |\omega| \leq \pi/T_{\text{max}} \\ 4W_d(\omega \tau_{\text{min}}) & |\omega| > \pi/T_{\text{max}} \end{cases} \quad (23)$$

The windowed stability variance bound is therefore

$$\sigma_{\text{sw}}^2(T, \tau) \leq \sigma_{\text{sw}}^2(T_{\text{max}}, \tau_{\text{min}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W'_{\text{sw}}(\omega T_{\text{max}}, \omega \tau_{\text{min}}) d\omega. \quad (24)$$

Thus, the over-bounding weighting function gives the lowest low-frequency response (at the largest T) and the highest high-frequency response (at the smallest τ).

Equation (23) was obtained by noting that the stability weighting function envelopes the windowed stability weighting function at frequencies below π/T_{max} . Four times the displacement weighting function envelopes the windowed stability weighting function at frequencies above π/T_{max} . If it cannot be shown that the stability requirement is satisfied with this formula, then the formula in (21) will have to be evaluated over a grid of T and τ for which the maximum is compared against the requirement.

Note that $\sigma_{\text{sw}}^2(T, \tau) \leq 4(\sigma_d^2 + \eta^2) \leq 4(\sigma_a^2 + \eta^2)$, so a windowed stability requirement is conservatively satisfied if $2(\sigma_d^2 + \eta^2)^{1/2}$ or $2(\sigma_a^2 + \eta^2)^{1/2}$ is less than the stability requirement. This test is less conservative if η is omitted.

An example of an over-bounding function on a variable windowed stability weighting function is shown in Figure 6. In this example, the SECCHI instrument on STEREO [7] has a variable exposure time (τ) from 0.1 to 1.0 second and the time between the center-of-integration of two subsequent exposures is $T_s = \tau + T_{\text{ro}}$ seconds, where the readout time T_{ro} is either 2.3 or 4.6 seconds, depending on whether the CCD chosen is 1024×1024 pixels or 2048×2048 pixels. In addition, three sequential images of different polarizations are coregistered, so the time between the center-of-integration of the first and third exposures is $T_s = 2(\tau + T_{\text{ro}})$ seconds. This defines four variable stability windows, $T_{s1}, T_{s2}, T_{s3}, T_{s4}$. The over-bounding weighting function is defined by $\tau_{\text{min}} = 0.1$ and $T_{\text{max}} = 25.2$ in equation (23). The various curves below the bound in Figure 6 are the four windowed stability weighting functions for $\tau = 0.1, 0.6$, and 1.0 seconds.

Coherence Time

It is shown in [2, Appendix B] that the coherence time τ_c of the jitter motion has to be small compared to the window time so that the time averages or expectation operations are valid. The coherence time is given by

$$\tau_c = [-2R_j(0)/\ddot{R}_j(0)]^{1/2} \quad (25)$$

where

$$R_j(\tau) = \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_j(\omega T) \exp(j\omega \tau) d\omega, \quad (26a)$$

$$\ddot{R}_j(0) = \int_{-\infty}^{\infty} \omega^2 S_{\Theta}(\omega) W_j(\omega T) d\omega. \quad (26b)$$

For line of sight motion that is dominated by narrow-band or sinusoidal frequencies, it is not necessary that τ_c be small but there has to be several cycles of motion within the window. The density function of narrow-band motion is not Gaussian, but an equivalent PSF can still be derived. This is given by [2, equation (B7)]. A Gaussian PDF can be assumed if there are several uncorrelated sinusoidal components in the motion.

Table 1: Accuracy, Displacement, Jitter, and Stability Metrics

Metric	Time Domain		Frequency Domain	
Accuracy	$\sigma_a^2 + \mu^2$	$= \mathcal{E}\{\Theta^2(t)\}$	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) \, d\omega$	
Displacement	$\sigma_d^2(T) + \mu^2$	$= \mathcal{E}\{\langle \Theta(t) \rangle_T^2\}$	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_d(\omega T) \, d\omega$	$T = T_d$
Jitter	$\sigma_j^2(T)$	$= \mathcal{E}\{\langle \Theta^2(t) \rangle_T - \langle \Theta(t) \rangle_T^2\}$	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_j(\omega T) \, d\omega$	$T = T_j$
Stability	$\sigma_s^2(T)$	$= \mathcal{E}\{(\Theta(t) - \Theta(t - T))^2\}$	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_s(\omega T) \, d\omega$	$T = T_s$
W-Stability	$\sigma_{sw}^2(T, \tau)$	$= \mathcal{E}\{(\bar{\Theta}(t) - \bar{\Theta}(t - T))^2\}$	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\Theta}(\omega) W_{sw}(\omega T, \omega \tau) \, d\omega$	$T = T_s$
	$W_d(\nu) = 2(1 - \cos \nu)/\nu^2$	displacement weighting function		
	$W_j(\nu) = 1 - 2(1 - \cos \nu)/\nu^2$	jitter weighting function		
	$W_s(\nu) = 2(1 - \cos \nu)$	stability weighting function		
	$W_{sw}(\nu) = W_s(\omega T) W_d(\omega \tau)$	windowed stability weighting function, $\tau = T_j$ (in general)		
	$\nu = \omega T$	normalized frequency ($T = T_d, T_j, T_s$ accordingly)		
	$\Theta(t)$ = attitude error in a given axis			
	$\bar{\Theta}(t) = \langle \Theta^2(t) \rangle_{\tau}$ = average attitude error over interval τ			
	$S_{\Theta}(\omega)$ = power spectral density of $\Theta(t)$			
	$\mathcal{E}\{\cdot\}$ = statistical expectation			
	$\langle \cdot \rangle_T$ = time average over a time window of width T			

Summary of Pointing Metrics

The pointing metrics given in the previous sections are summarized in Table 1. The window T is in general different for the displacement, jitter, and stability metrics. As indicated in Table 1, these and the accuracy metric can be computed in either the time domain or in the frequency domain. The expressions are mathematically equivalent; there is no ambiguity in their relationship. Because these are mean-square (MS) quantities, they are easily computed by using standard control system analysis methods. For sampled data, these methods include the sample mean and variance and the Fast Fourier Transform (FFT).

Pointing Performance Computation

Computation of the pointing metrics is most conveniently performed in the frequency domain even if the attitude error is produced in the time-domain by a time-domain simulator. In the frequency domain, the main tool for computing the pointing metrics from uniformly sampled data

is the Fast Fourier Transform (FFT). The FFT is scaled to a power spectrum (not a density) by dividing it by M and then computing its magnitude squared, where M is the number of samples of data and is assumed to be a power of two. The power spectrum is then shifted (FFTSHIFT in Matlab) so that the zero frequency line is at the center. The frequencies then range from $-(M/2)/M\delta$ to $(M/2-1)/M\delta$ in increments of $1/M\delta$ Hz, where δ is the sample interval. Note that the sum of the discrete power spectrum is equal to the second moment of the data, $(1/M) \sum_1^M x_i^2$ (which equals the variance plus the mean squared).

In the frequency domain, the pointing metrics are evaluated by computing the weighting functions at each frequency point, multiplying the power spectrum of the attitude error by the weighting functions, and then summing the terms. This result may be computed by considering only non-negative frequencies, but the zero-frequency term has to be multiplied by one and the positive-frequency terms multiplied by two. The computational algorithms in the frequency domain are summarized in Table 2. An upper bound on the windowed stability variance can be calculated by

$$\sigma_{\text{sw}}^2(T_s, T_j) \leq \sigma_{\text{sw}}^2(T_{s,\text{max}}, T_{j,\text{min}}) = \sum_{i=1}^M P(\omega_i) W'_{\text{sw}}(\omega_i T_{s,\text{max}}, \omega_i T_{j,\text{min}}) \quad (27)$$

where $W'_{\text{sw}}(\omega T_{s,\text{max}}, \omega T_{j,\text{min}})$ is given in (23) with $\tau_{\text{min}} = T_{j,\text{min}}$.

EXAMPLE: BACK-OF-THE-ENVELOPE CALCULATION

Although the computation of the displacement, jitter, and stability metrics may seem formidable, in reality it is quite simple and can be used for preliminary design analysis. This is illustrated by the following “back-of-the-envelope calculation” of jitter.

Suppose we have a disturbance at n frequencies due to reaction wheel imbalance. This disturbance can be modeled by

$$d(t) = \sum_{k=1}^n A_k \sin \omega_k t, \quad A_k = (U_d + r U_s) \omega_k^2 \quad (28)$$

where U_d and U_s are the dynamic and static imbalance of the wheels and r is the distance of the wheel from the center of mass. The power spectral density (PSD) of the disturbance is

$$|D(j\omega)|^2 = \sum_{k=1}^n \frac{A_k^2}{2} [\delta(\omega + \omega_k) + \delta(\omega - \omega_k)] \quad (29)$$

where

$$\delta(x) = \begin{cases} \pi & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is the Dirac delta function with area π . For disturbance frequencies that are outside the bandwidth of the attitude controller, the disturbance-to-attitude transfer function is simply $H(s) = 1/Js^2$, where J is the spacecraft inertia about a given axis and $s = j\omega$ is complex frequency. The attitude error spectrum is then $S_\Theta(\omega) = |H(j\omega)|^2 |D(j\omega)|^2$. By substituting these into (11), we obtain the

Table 2: Algorithms to compute the accuracy, displacement, jitter, and stability metrics

Attitude Error Metric	$\sigma_a^2 + \mu^2 = \sum_{i=1}^M P(\omega_i)$
Displacement Metric	$\sigma_d^2(T_d) + \mu^2 = \sum_{i=1}^M P(\omega_i) W_d(\omega_i T_d)$
Jitter Metric	$\sigma_j^2(T_j) = \sum_{i=1}^M P(\omega_i) W_j(\omega_i T_j)$
Stability Metric	$\sigma_s^2(T_s) = \sum_{i=1}^M P(\omega_i) W_s(\omega_i T_s)$
Windowed Stability Metric	$\sigma_s^2(T_s, T_j) = \sum_{i=1}^M P(\omega_i) W_s(\omega_i T_s) W_d(\omega_i T_j)$

$M = 2^n$	length of data record
$\omega = 2\pi[-M/2 : 1 : M/2 - 1]/M\delta$	frequency range (rad/sec), δ = sample time (sec)
$P(\omega) = \text{FFTSHIFT}(\text{FFT}(\Theta, M)) ^2/M$	power spectrum of Θ
$W_d(\nu) = 2(1 - \cos \nu)/\nu^2$	displacement weighting function, $\nu = \omega T_d$
$W_j(\nu) = 1 - 2(1 - \cos \nu)/\nu^2$	jitter weighting function, $\nu = \omega T_j$
$W_s(\nu) = 2(1 - \cos \nu)$	stability weighting function, $\nu = \omega T_s$

jitter variance as simply

$$\begin{aligned}
\sigma_j^2(T) &= \sum_{k=1}^n \frac{A_k^2}{2} |H(j\omega_k)|^2 W_j(\omega_k T) \\
&= \sum_{k=1}^n \frac{1}{2} ((U_d + rU_s)\omega_k^2)^2 \left(\frac{1}{J\omega_k^2}\right)^2 \left(1 - \frac{2(1 - \cos \omega_k T)}{(\omega_k T)^2}\right) \\
&= \frac{1}{2} ((U_d + rU_s)/J)^2 \sum_{k=1}^n \left(1 - \frac{2(1 - \cos \omega_k T)}{(\omega_k T)^2}\right)
\end{aligned} \tag{30}$$

Advantage was taken of the fact that the integrand is an even function, so the summation is only over positive frequencies. Stability and displacement can be evaluated similarly by substituting the corresponding weighting functions. Flexible-body modes can be treated by assuming some modal gain, *e.g.*, by increasing one or more of the A_k by 30 or 40 dB. By omitting structural modes and other details, the closed-loop system $H(s)$ may be modeled as a third-order transfer function from the disturbance input to the attitude error output. The transfer function of the closed-loop attitude

control system is of the form

$$H(s) = \frac{ks}{(s + a)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

where $-a$ is the real pole, ζ is the damping, ω_n is the natural frequency, and k a gain. The transfer function is third order when proportional-integral-derivative control is used. It may be approximated as second order since the real pole should not be dominant.

It is shown in [10] that the weighting functions can be approximated by rational transfer functions. A plant model can then be augmented with these transfer functions and a controller can be synthesized to minimize an H_2 -norm or mixed H_∞/H_2 -norm optimality criterion so that the various pointing requirements are met. For an existing controller design, the plant can be augmented with the rational approximations and the response to sensor and actuator noise can be determined by solving a Lyapunov equation (via the LYAP function in Matlab).

POINTING SPECIFICATIONS

1 σ Versus 3 σ Requirements

Pointing requirements are most often specified as 3 σ values with the implication or overt statement being that the error must be less than the given value “99.7% of the time” or “almost always”. These are often stringent requirements with little connection to actual instrument performance. The jitter definition in this paper is a 1 σ value specified in connection with the point spread function (PSF) of the optics in an instrument [2] and the pixel size. The PSF and the pixel resolution are key factors in determining image quality.

Those who write specifications should be aware that for Gaussian error, multiplying the standard deviation by 3 creates a minor inconvenience for the attitude control engineer because control system engineers work with variances and standard deviations. Specifications have to be divided by 3 upon input to an analysis program and results have to be multiplied by 3 before they are reported. This silly practice of specifying 3 σ values rather than 1 σ values hopefully will now change.

Linear Error Versus Circular Error Specifications

Since resolution is generally defined in a single direction on a focal plane [2], jitter and stability specifications are usually single-axis specifications. In some cases it makes sense to state a radial error requirement, *e.g.*, antenna pointing. Under the assumption that the linear (single-axis) pointing error is Gaussian, the radial error is Rayleigh distributed with variance equal to the linear variance. The probability of exceeding 1 σ radially is 1%, whereas the probability of exceeding 1 σ along one axis is 0.27%. Since the probability is chosen arbitrarily (“almost always”), it can be argued that the linear specification should not be reduced to make the probability of a 3 σ radial error equal to the single-axis 3 σ probability of 0.27%.

Specification for Line-of-Sight Rotation of an Imaging Instrument

For rotation about the LOS (sometimes called roll or boresight), the pointing error at the edge of a focal plane is reduced by $1/\tan(\frac{1}{2}\text{FOV}) \simeq 2/\text{FOV}$, where FOV is the field of view in radians. Therefore the boresight pointing error requirement for an imaging instrument should be 2/FOV times the cross-boresight pointing requirement. This generally would apply only to jitter and stability because an offset in the orientation of the image is usually, but not always, unimportant.

CONCLUSION

Pointing error definitions and metrics for accuracy, displacement, and jitter were re-introduced in this paper and new definitions and metrics called stability and windowed-stability were introduced. These pointing error metrics are related directly to image quality and are also useful for other types of measurements. The pointing error definitions and metrics are unambiguous and mathematically well defined with equivalent mathematical descriptions in the time domain and in the frequency domain. The frequency-domain formulae are particularly easy to apply and are propitious in attitude control system design and analysis. It is hoped that the pointing error definitions and metrics will be adopted as a standard within NASA and industry for specification and verification of requirements.

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